Variational Approach to the Problem of Coupled Electromagnetic Wave-Guides

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(October 27, 1997)

In the following text we derive a set of simple differential equations describing the field exchange between N straight mutually parallel light waveguides arranged in an array. We apply a variational analysis where eigenmodes of the individual fibers in absence of other members of the array are used as "seed" functions. We point out that equations we derived contain some "counter-intuitive" terms usually overlooked in a heuristic consideration.

I. VARIATIONAL FORMULATION FOR THE TIME-INDEPENDENT MAXWELL EQUATIONS IN AN INHOMOGENEOUS ISOTROPIC MEDIUM

Theorem 1. Solution of a set of Maxwell equations

$$\begin{aligned} [\vec{\nabla} \times \vec{E}(\vec{r})] &= -i\omega\mu(\vec{r})\vec{H}(\vec{r}) \\ [\vec{\nabla} \times \vec{H}(\vec{r})] &= +i\omega\epsilon(\vec{r})\vec{E}(\vec{r}) \\ &\vec{r} \in \mathcal{V} \end{aligned} \tag{1}$$

obeying boundary conditions

$$\begin{aligned} \vec{E}(\vec{r}) &= \vec{E}_{\sigma_1}(\vec{r}) \; : \; \vec{r} \in \sigma_1 \subset \sigma \\ \vec{H}(\vec{r}) &= \vec{H}_{\sigma_1}(\vec{r}) \; : \; \vec{r} \in \sigma_1 \subset \sigma \end{aligned} \tag{2}$$

(σ_1 is a part of the surface σ such that for any distribution of the fields $\vec{E}_\sigma$, $\vec{H}_\sigma$ on the boundary σ_1 the Maxwell equations (1) have one and only one solution, σ is the surface of the volume $\mathcal{V}$ where the fields are defined) is an extremum of a functional

$$\begin{aligned} &S([Re(\vec{E})],\, [Im(\vec{E})],\, [Re(\vec{H})],\, [Im(\vec{H})]) \\ &= \int_{\mathcal{V}} d\mathcal{V} \left\{ \omega\epsilon\vec{\mathcal{E}}^*\vec{\mathcal{E}} + \omega\mu\vec{\mathcal{H}}^*\vec{\mathcal{H}} \right. \\ &\quad + \frac{i}{2}\Big[\vec{E}^*[\vec{\nabla} \times \vec{H}] - \vec{H}^*[\vec{\nabla} \times \vec{E}] \qquad (3) \\ &\qquad \left. -\vec{E}[\vec{\nabla} \times \vec{H}^*] + \vec{H}[\vec{\nabla} \times \vec{E}^*]\Big] \right\} \qquad (4) \end{aligned}$$

on a class of the bi-vector fields $(\vec{E}(\vec{r}),\, \vec{H}(\vec{r}))$ such that they

(a) satisfy the boundary conditions (2);

(b) satisfy some compatible with (2) boundary conditions

$$\begin{aligned} \vec{E}(\vec{r}) &= \vec{E}_{\sigma/\sigma_1}(\vec{r}) \; : \; \vec{r} \in \sigma/\sigma_1 \\ \vec{H}(\vec{r}) &= \vec{H}_{\sigma/\sigma_1}(\vec{r}) \; : \; \vec{r} \in \sigma/\sigma_1 \end{aligned} \tag{5}$$

on the complementary surface σ/σ_1;

(c) have a curl everywhere within the volume $\mathcal{V}$ [1].

Proof of the Theorem 1 is based on a straightforward evaluation of the extremum condition

$$\delta S = 0 \; : \; \forall\, [\delta Re(\vec{E})],\, [\delta Im(\vec{E})],\, [\delta Re(\vec{H})],\, [\delta Im(\vec{H})]. \tag{6}$$

The terms of the form $\vec{\mathcal{A}}[\vec{\nabla} \times \delta\vec{\mathcal{B}}]$ should be evaluated as

$$\begin{aligned} &\int_{\mathcal{V}} d\mathcal{V}\, \vec{A}[\vec{\nabla} \times \delta\vec{B}] \\ &\qquad = \textstyle\int_{\mathcal{V}} d\mathcal{V}\, \delta\vec{B}[\vec{\nabla} \times \vec{A}] + \int_\sigma d\vec{\sigma}\, \vec{\nabla}[\delta\vec{B} \times \vec{A}] \end{aligned} \tag{7}$$

After that the surface terms should be disregarded according to the conditions (a) and (b).

II. PARALLEL WAVE-GUIDES PROBLEM. VARIATIONAL APPROACH

Consider a nonmagnetic translationally invariant medium

$$\begin{aligned} \epsilon &= \bar{\epsilon}(x,y) + \sum_k \Delta\epsilon_k(x,y)\,; \\ \mu &= \mu_0 \end{aligned} \tag{8}$$

We suppose also that any elementary medium $\bar{\epsilon} + \Delta\epsilon_k$ forms a single-mode waveguide and the "bar" medium $\bar{\epsilon}$ does not support any propagating solutions at all. We are looking for a solution of the Maxwell equations defined in a stripe $\mathcal{V}$

$$\begin{aligned} -\infty < x &< +\infty\,; \\ -\infty < y &< +\infty\,; \\ 0 \le z &\le z_{max} \end{aligned} \tag{9}$$

with some boundary conditions

[1] The fields may still be discontinuous. Only the cross-derivatives (e.g. $\partial E_x/\partial y$) must be well defined.

$$\vec{E}(x,y,0) = \vec{E}^{Init}(x,y)$$
$$\vec{H}(x,y,0) = \vec{H}^{Init}(x,y) \qquad (10)$$

defined on the plane $\sigma_1 = \{-\infty < x < +\infty,\ -\infty < y < +\infty,\ z = 0\}$.

The idea of our approach is to approximate solution of the boundary conditions problem (1,8,10) with a sum

$$\begin{pmatrix} \vec{E}(x,y,z) \\ \vec{H}(x,y,z) \end{pmatrix} \approx \sum_k a_k(z) \begin{pmatrix} \vec{E}^0_k(x,y) \\ \vec{H}^0_k(x,y) \end{pmatrix} \qquad (11)$$

where

$$\begin{pmatrix} \vec{E}_k(x,y,z) \\ \vec{H}_k(x,y,z) \end{pmatrix} = \begin{pmatrix} \vec{E}^0_k(x,y) \\ \vec{H}^0_k(x,y) \end{pmatrix} \exp(i\beta_k z) \qquad (12)$$

is an *exact* solution of the reduced Maxwell equations

$$[\vec{\nabla} \times \vec{E}_k(\vec{r})] = -i\omega\mu_0 \vec{H}_k(\vec{r})$$
$$[\vec{\nabla} \times \vec{H}_k(\vec{r})] = +i\omega(\bar{\epsilon} + \Delta\epsilon_k)\vec{E}_k(\vec{r}) \qquad (13)$$

characterized by a longitudinal wave number β_k

We postulate that the best approximation to the "true" solution of the Maxwell equations (1,8,10)[2] is an extremum of a reduced action

$$S_{\text{reduced}}([Re(a)],\ [Im(a)]) \qquad (14)$$
$$= S([Re(\vec{E}([a]))],\ [Im(\vec{E}([a\ ]))],$$
$$[Re(\vec{H}([a]))],\ [Im(\vec{H}([a]))])$$

($[a]$ is a set of a_k) on the space of functions $[a](z)$ such that they

(a) satisfy some initial conditions

$$a_k(0) = a_k^{Init}; \qquad (15)$$

(b) satisfy some compatible with (15) final conditions

$$a_k(z_{max}) = a_k^{z_{max}}; \qquad (16)$$

(c) have a first derivative everywhere within the stripe $[0;\ z_{max}]$.

A straightforward extremization of the functional (14) leads to the *reduced field equation* for the coefficients a_k

$$i \sum_{k'} P_{k,k'}\, \partial_z a_{k'} = \sum_{k''} H_{k,k''} a_{k''}\ , \qquad (17)$$

[2]It must be clarified what "the best" means, *i.e.* with respect to which metrics. This is a subject of a separate research. An only excuse for the disregarding of this point is that it is also usually disregarded in Quantum Mechanics either.

where

$$H_{k,k'} = \int dx\, dy \qquad (18)$$
$$\{\ \omega\delta\epsilon_{k,k'} (\vec{E}^0_k)^* \vec{E}^0_{k'} - \frac{1}{2}(\beta_k P_{k,k'} + P_{k,k'}\beta_{k'})\ \}\ , \qquad (19)$$

$$P_{k,k'} = \int dx\, dy \qquad (20)$$
$$\vec{e}_z \cdot \{\ [(\vec{E}^0_k)^* \times \vec{H}^0_{k'}] + [\vec{E}^0_{k'} \times (\vec{H}^0_k)^*]\ \}\ ,$$

and

$$\delta\epsilon_{k,k'} = \begin{cases} \frac{1}{2}(\Delta\epsilon_k + \Delta\epsilon_{k'}) + \sum_{k''\neq k,k'} \Delta\epsilon_{k''} & : \ k \neq k' \\ \sum_{k''\neq k} \Delta\epsilon_{k''} & : \ k = k' \end{cases} \qquad (21)$$

Notice that the variational procedure, we presented, does not tell anything about how to choose the initial conditions for the coefficients a_k. In the next section we present some intuitive arguments how to do that. A rigorous procedure will involve somewhat deeper understanding of the question on with respect to which metrics the equations we got are optimal.

III. FORMAL ANALYSIS OF THE REDUCED FIELD EQUATIONS

After some lengthy but straightforward algebra one may show that the equations (17) can be obtained by a projection of the Maxwell equations (1,8) to a linear subspace spanned by our "seed" modes

$$\langle \chi_k | \hat{\mathcal{L}}_{\text{Maxwell}} \sum_{k'} a_{k'}(z) | \chi_{k'} \rangle = 0\ :\ \forall k\ , \qquad (22)$$

where

$$|\chi\rangle = \begin{pmatrix} \vec{E}^0(x,y) \\ \vec{H}^0(x,y) \end{pmatrix} \qquad (23)$$

is a complex bi-vector field defined on the plane XY,

$$\hat{\mathcal{L}}_{\text{Maxwell}} = \begin{pmatrix} \omega\epsilon & +i\vec{\nabla}\times \\ -i\vec{\nabla}\times & \omega\mu \end{pmatrix} \qquad (24)$$

is the Maxwell Lagrangian, and the scalar product is defined as

$$\langle \chi_1 | \chi_2 \rangle = \int dx\, dy\ \{\ (\vec{E}^0_1)^* \vec{E}^0_2 + (\vec{H}^0_2)^* \vec{H}^0_1\ \}\ , \qquad (25)$$

Notice that the reduced field equations (17) preserve a hermitian quadratic form

$$\partial_z \prec a_{(1)}|a_{(2)} \succ = 0$$

$$\prec a_{(1)}|a_{(2)} \succ = \sum_{k,k'} a^*_{(1),k} P_{k,k'} a_{(2),k'} \qquad (26)$$

which is a projection of a more general form

$$\prec \chi_1|\chi_2 \succ = \int dx\, dy \vec{e}_z \cdot \{\, [(\vec{E}^0_1)^* \times \vec{H}^0_2] + [\vec{E}^0_2 \times (\vec{H}^0_1)^*]\,\}\,, \qquad (27)$$

to our reduced subspace. The conservation law (26) shows, in particular, that our reduced field equations preserve the energy flux

$$P_z = \int dx\, dy \vec{e}_z \cdot \{\, [(\vec{E})^* \times \vec{H}] + [\vec{E} \times (\vec{H})^*]\,\}\,, \qquad (28)$$

along the wave guide. Note that the quadratic form (26) (and the corresponding matrix $P_{k,k'}$) is not positively defined and, therefore, may not be a "good" scalar product. Below we will show that for the case of a weak coupling between the wave guides the form (26) do be positively defined that greatly simplifies the field equations.

The projection procedure (22) suggests that the correct initial conditions for the reduced field equations (17) are

$$a^{Init}_k = \sum_{k'} Q^{-1}_{k,k'} \langle \chi_k | \chi^{Init} \rangle \qquad (29)$$

where

$$Q_{k,k'} = \langle \chi_k | \chi_{k'} \rangle \qquad (30)$$

IV. FURTHER SIMPLIFICATION FOR THE CASE OF WEAK COUPLING

From now on we suppose that our modes are normalized to the "unit energy flux"

$$\prec \chi_k|\chi_k \succ = 1 \,:\, \forall k \qquad (31)$$

This means, in particular that all the modes are propagating "downstream" the wave guides from 0 to z_{Max} In this case we may prove the following

<u>Lemma 1</u> The hermitian quadratic form (26) defined by the matrix $P_{k,k'}$ is positively defined if the coupling is weak enough

$$|\prec \chi_k|\chi_{k'} \succ| < \varepsilon \ll 1 \,:\, \forall k \neq k' \qquad (32)$$

To prove the Lemma we should notice that in the case of zero coupling the matrix $P_{k,k'}$ is equal to the unit matrix. An important consequence of the Lemma 1 is the existence the inverse square root of the matrix $P_{k,k'}$.

Consider the following transformation of the field coordinates

$$a_k = \sum_{k'} P^{-1/2}_{k,k'}\, b_{k'} \qquad (33)$$

Such a transformation leads to an equation of hermitian form

$$i\,\partial_z b_{k'} = \sum_{k'} \mathcal{H}_{k,k'}\, b_{k'}\,, \qquad (34)$$

where

$$\mathcal{H} = P^{-1/2} H P^{-1/2} \qquad (35)$$

Further simplification arouses if we present the matrices H an P in a form

$$\begin{aligned} H &= H_0 + \Delta H + \frac{1}{2} H_0 \Delta P + \frac{1}{2} \Delta P H_0 \\ P &= I + \Delta P \end{aligned} \qquad (36)$$

where

$$\begin{aligned} (H_0)_{k,k'} &= -\beta_k \delta_{k,k'} \\ (\Delta H)_{k,k'} &= \int dx\, dy\, \omega \delta\epsilon_{k,k'} (\vec{E}^0_k)^* \vec{E}^0_{k'} \\ I_{k,k'} &= \delta_{k,k'} \\ (\Delta P)_{k,k'} &= \begin{cases} 0 & : \; k = k' \\ \prec \chi_k|\chi_{k'} \succ & : \; k \neq k' \end{cases} \end{aligned} \qquad (37)$$

an then treat the "Δ'ed" operators to be small to the same order. In this case the "Hamiltonian" $\mathcal{H}$ reduces to a simple expression

$$\mathcal{H} \approx H_0 + \Delta H \qquad (38)$$

which can be used to describe the field dynamics.

V. FORMULAIRE

In this section we will summarize the results.

In the case of a weak coupling the field exchange between the wave guides is described by an "Schrödinger equation"

$$i\,\partial_z b_{k'} = \sum_{k'} \mathcal{H}_{k,k'}\, b_{k'}\,, \qquad (39)$$

where

$$\begin{aligned} \mathcal{H} &\approx H_0 + \Delta H \\ (H_0)_{k,k'} &= -\beta_k \delta_{k,k'} \\ (\Delta H)_{k,k'} &= \int dx\, dy\, \omega \delta\epsilon_{k,k'} (\vec{E}^0_k)^* \vec{E}^0_{k'} \end{aligned} \qquad (40)$$

and[3]

$$\delta\epsilon_{k,k'} = \begin{cases} \frac{1}{2}(\Delta\epsilon_k + \Delta\epsilon_{k'}) + \sum_{k''\neq k,k'} \Delta\epsilon_{k''} \ , & : \ k \neq k' \\ \sum_{k''\neq k} \Delta\epsilon_{k''} & : \ k = k' \end{cases} \quad (41)$$

The electromagnetic field is supposed to be expanded to a sum over the individual waveguide modes

$$\begin{pmatrix} \vec{E}(x,y,z) \\ \vec{H}(x,y,z) \end{pmatrix} \approx \sum_k a_k(z) \begin{pmatrix} \vec{E}_k^0(x,y) \\ \vec{H}_k^0(x,y) \end{pmatrix} \quad (42)$$

where

$$a_k = \sum_{k'} P_{k,k'}^{-1/2}\, b_{k'} \ , \quad (43)$$

$$P_{k,k'}^{-1/2} \approx \delta_{k,k'} - \frac{1}{2}(1-\delta_{k,k'}) \prec \chi_k|\chi_{k'} \succ \quad (44)$$

and the definition of the quadratic form $\prec \cdot|\cdot \succ$ is given in (26). Initial conditions for the coefficients a_k are given in (29), where the scalar product $\langle\cdot|\cdot\rangle$ is defined in (25).

Note that in the case of a linear waveguide array the matrix elements of ΔH can be presented in an alternative form:

$$\begin{aligned}(\Delta H)_{k,k'} &\approx \delta_{|k'-k|,1} \int dx\, dy\, \omega \frac{1}{2}(\Delta\epsilon_k + \Delta\epsilon_{k'})(\vec{E}_k^0)^* \vec{E}_{k'}^0 \\ &= -i \int_{l_{k,k'}} d\vec{l}\, p_{k,k'} \\ &\quad -(\beta_k - \beta_{k'}) \left(\int_{\sigma_k} - \int_{\sigma_{k'}} \right) dx\, dy\, \vec{e}_z \cdot p_{k,k'} \quad (45)\end{aligned}$$

where

$$p_{k,k'}(x,y) = \quad (46)$$

$$\vec{e}_z \ \cdot \ \{ \, [(\vec{E}_k^0)^* \times \vec{H}_{k'}^0] + [\vec{E}_{k'}^0 \times (\vec{H}_k^0)^*] \, \} \quad (47)$$

are the "matrix elements" of the energy flux density, the plane XY is split by a line $l_{k,k'}$ to two parts σ_k and $\sigma_{k'}$ containing the k's and k''s waveguide respectively. The differential element of the line $d\vec{l}$ is oriented from the k's domain to the k''s domain and is orthogonal to the line. The last term is usually disregarded in literature, though it vanishes only in the degenerate case $\beta_k = \beta = const(k)$. Note that the formula (45) takes into account coupling between the neighboring waveguides only and neglects all other couplings as being exponentially small. The expression (45) is similar to the analogous Quantum Mechanics expression for the Josephson coupling, where the energy flux is replaced by the particle flux.

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[3]Remind that due to an unusual for Quantum Mechanics convention $\vec{E}(\vec{r},t) = \vec{E}(\vec{r})\exp(+i\omega t) + c.c.$ the wave propagating downstream the waveguide corresponds to negative values of the wavevector β.