Is Boltzmann Entropy Time's Arrow's Archer?

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Ludwig Boltzmann's ideas on irreversibility are as controversial today as they were at their introduction a hundred years ago. In the article "Boltzmann's Entropy and Time's Arrow" (September 1993, page 32), Joel Lebowitz, by giving a modern exposition of Boltzmann's ideas, tries to assure us that the controversy is unwarranted. Readers left unpersuaded should know that they are not alone. Boltzmann's ideas are indeed controversial, because Boltzmann failed to palce them on a firm conceptual foundation. Today a firm foundation can be provided—the key ideas are Claude Shannon's statistical information [1] and Edwin Jaynes's principle of maximum entropy [2]—but Lebowitz's update, instead of providing the necessary clarification, recapitulates the same murky concepts in modern language.

Lebowitz addresses how time-asymmetric behavior of macroscopic variables arises from time-symmetric microscopic equations. He partitions phase space into macrostates, coarse-grained cells M_i (of phase-space volume $|\Gamma_{M_i}|$) defined by the values of the macroscopic variables of interest—for example, the numbers of particles within identical cubes that fill configuration space. To each phase-space point, or microstate, in M_i he assigns the Boltzmann entropy $S_{\mathrm{B}}(M_i) = k \log |\Gamma_{M_i}|$. If the system is initially confined to a small phase-space cell, then when the constraints are released, it will tend to wander into larger cells. Lebowitz quantifies this behavior in terms of the Boltzmann entropy, which tends to increase along a "typical" trajectory.

The problem here is not the story so much as the commentary; for someone outlining an avowedly statistical theory, Lebowitz betrays an odd mistrust of probability concepts. He stresses that he is dealing with the typical behavior of individual systems, not with average behavior within an ensemble. But how can one characterize typical behavior without reference to a probability distribution? Furthermore, he dismisses the Gibbs entropy $S_{\mathrm{G}} = -k \int d\Gamma \rho \log \rho$ of a phase-space probability distribution ρ as irrelevant to nonequilibrium phenomena, partly because it remains constant under Hamiltonian evolution, but also because it relies on probabilities. Yet what is the significance of the increase of the Boltzmann entropy when it has an interpretation as a physical quantity only in thermodynamic equilibrium? Indeed, why attribute a Boltzmann entropy to each phase-space point when the Boltzmann entropy is wholly a property of the coarse-graining?

Dealing with these questions entails using probabilities. Lebowitz implies that probabilistic predictions apply only to physical ensembles. To the contrary, when probabilities are sharply peaked, as they are for certain macroscopic variables, they make reliable predictions for *individual* systems. Probabilities provide the *only* way to define typical behavior for individual systems and to assess just how typical it is.

The phase-space probability distribution $\rho(t)$ at time t follows from applying the system dynamics to a uniform distribution on the initial cell. The statistics of the macroscopic variables at time t, determined by the probabilities $p_i(t) = \int_{M_i} d\Gamma\, \rho(t)$ to be in cell M_i, are unaffected if $\rho(t)$ is replaced, within each cell M_i, by a uniform probability distribution containing probability $p_i(t)$. This coarse-grained phase-space distribution can be characterized uniquely as having the maximum Gibbs entropy given the probabilities $p_i(t)$, the maximum being $\bar{S}_{\rm G} = -k\sum_i p_i \log p_i + \sum_i p_i S_{\rm B}(M_i)$.

Lebowitz's insistence on the primacy of Boltzmann entropy over Gibbs entropy is thus stood on its head. The Gibbs entropy $\bar{S}_{\rm G}$ of the coarse-grained distribution generally increases. Moreover, the increase has a compelling interpretation: Since $S_{\rm G}/k$ is Shannon's statistical information, the difference between $\bar{S}_{\rm G}$ and the initial Gibbs entropy is the amount of information discarded when one retains only the statistics of the macroscopic variables. The *average* Boltzmann entropy does contribute to $\bar{S}_{\rm G}$, but this appearance of the Boltzmann entropies has nothing to do with entropies of individual phase-space points; rather, it is a direct expression of having discarded all information about the details of $\rho(t)$ within the coarse-grained cells.

As Jaynes has emphasized [2], firm conceptual foundations are required for progress in physics. The shaky foundations provided by Boltzmann and Lebowitz obscure both what has been accomplished and what remains to be done. Boltzmann's ideas can indeed by used to derive time-asymmetric equations for macroscopic variables, once they are supported within the solid framework of Gibbs, Shannon and Jaynes; the Gibbs entropy $\bar{S}_{\rm G}$ *explains* the time asymmetry as a consequence of discarding microscopic information that is unnecessary for predicting the behavior of the macroscopic variables. Yet this explanation, like all good ones, immediately raises other questions: Why coarse-grain? Why discard information? These questions, the true puzzles of irreversibility, provide the arena for further work [3].

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[1] C. E. Shannon, W. Weaver, *The Mathematical Theory of Communication*, U. of Illinois P., Urbana (1949).

[2] E. T. Jaynes, *Papers on Probability, Statistics and Statistical Physics*, R. D. Rosenkrantz, ed., Kluwer, Dordrecht, The Netherlands (1983).

[3] W. H. Zurek, Phys. Rev. A **40**, 4731 (1989). C. M. Caves, Phys. Rev. E **47**, 4010 (1993).