Landauer's Principle and Black-Hole Entropy

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Abstract

Landauer's principle that the world's entropy increases by at least $k_B ln 2$ upon a one-bit erasure in a memory coupled to an infinite heat bath is generalized to the nominally different case of dumping information past a black hole's event horizon. Specifically this idea is used to provide a lower bound on the black-hole entropy to horizon area ratio using only a modicum of classical general relativity and relativistic quantum mechanics; quantum field theoretic methods are not required. The dumping of free 1-D scalar-particle wave packets leads to an estimate of .181 *for the lower bound of the said ratio. This compares well to the established value of exactly* 1/4 *for the ratio itself and contrasts with Bekenstein's original "information theoretic" estimate of* .028.

I. Introduction

Landauer's principle [1,2] is the thermodynamic accounting rule that requires a minimum free energy expenditure of $k_B T ln 2$ in the erasure of one bit of information by an engine coupled to an infinite heat bath at temperature T. Use of this principle has recently proven crucial to the correct analysis of the limitations of Maxwellian demons [3] and, moreover, led to Zurek's [4,5] proposal that physical entropy S consists not only of the standard statistical contribution $H = -k_B \sum p_i ln p_i$ but also an algorithmic information theoretic term $K(data)$. This term, Chaitin's algorithmic entropy/complexity [6], represents the length of the shortest self-delimiting code-word (with respect to some given universal Turing machine) for specifying a demon's data on the system's exact dynamic state. In part this proposal is allowed by means of the thermodynamic relation $F = E - TS$; for from this one can say that Landauer's principle entails that *a one-bit erasure is necessarily accompanied by a minimum one-bit (i.e. $k_B ln 2$) increase in the world's entropy.* This ingredient leading to Zurek's proposal may be termed the entropic formulation of Landauer's principle. In this report, Zurek's proposal is taken to be incontestable whenever such an entropic Landauer principle is valid.

Caution, however, is clearly required in the usage of this entropic formulation. One must not forget the limited physical situation in which it receives its justification—engines coupled to environments with specified temperatures T, *i.e.* infinite heat baths. One could, for instance, wonder how well this version of the principle fairs when the erasure comes about not by means of a coupling to an infinite heat bath but rather by some other mechanism, say by the coupling to a larger but finite environment with fixed energy E—the microcanonical ensemble? Or even whether there are at all any other means than the one already considered of true information erasure? Such questions must definitely be fleshed out in detail before the entropic Landauer principle can be attributed a universal validity. It is this train of thought that leads directly to black-hole physics.

At least one other independent mechanism for information erasure is apparently supplied by the "no hair" theorems of black hole physics [7]. In classical general relativity a black hole in equilibrium is characterized by nothing more than three parameters: mass, angular momentum, and electric charge. Consequently in the event that a physical information-bearing signal is captured by a black hole, most everything about that signal will be "erased" from the world. To take a specific instance, consider the various possible signals that may be sent by means of a beam of a given number of identical spins whose net angular momentum totals up to zero. The pattern of spin-up and spin-down slots in the signal might be used to convey a binary signal. In the approximation that the spins are noninteracting, one can easily imagine injection schemes in which each distinct signal leads nevertheless to the same identically parametered black hole. Hence the claim of erasure. Similar scenarios can be substituted for messages written in books of a given mass, messages encoded in the delay time between a fixed number of

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identical particle's arrivals at some receiver, or any of a countless number of other examples. Needless to say though, it is not as if everything else in the world excepting the disappearance of the signal remains constant during these capture processes; the black hole inevitably emerges with a new set of parameters and consequently a new event-horizon area. In particular, because of Hawking's area theorem [8], in realistic situations the event horizon area will always increase in a signal erasure.

These facts hint that the natural analog to an entropic Landauer principle in black-hole physics is perhaps simply a trade-off between information loss and area growth. Indeed it is already well known that black holes do actually have an associated entropy S with all the properties that one would expect of a thermodynamic entropy. Moreover this entropy is directly proportional to the event horizon area, *i.e.* $S = \eta A$ (setting $G = c = \hbar = k_B = 1$). This was first conjectured by Bekenstein in 1972, motivating it for the most part as a means of salvaging the Second Law of Thermodynamics in the world external to the horizon. Of particular relevance to the present purpose, it should be noted that Bekenstein's method for estimating the value of η [9], although slightly off base, involved certain "information theoretic" considerations that we shall later build upon. Subsequent workers have by various means (quantum field theory on curved backgrounds [10], quasi-statically building up a black hole quanta by quanta carefully taking into account quantum field theoretic acceleration radiation [11], path integral methods for thermodynamics [12], *etc.*) supplied the exact value of $\eta = 1/4$ for this relation.

The mission of this report should now be clear: to answer the question of how well the entropic Landauer principle fares in this situation so disparate from its original justification by explicitly checking how well it meshes with the established value for η. Section II below reviews the work of Bekenstein, shows how this hints at the signaling scheme most convenient for the present considerations, and lays out the plan of the calculation. Section III introduces the appropriate concepts from relativistic quantum mechanics and carries through the harder part of the work. Section IV declares the final estimate of $\eta \geq .181$ and closes with a small discussion.

II. The Implementation of Landauer's Principle

The precedent for using "information theoretic" considerations in estimating η was already set long ago by Bekenstein himself [9]. Briefly, his reasoning runs as follows. Consider a simple classical particle of rest mass m and proper radius r allowed to free fall past a black hole's event horizon. Depending upon the exact way the particle encounters the horizon, one can expect the horizon area to grow by various distinct amounts. It turns out, however, that there is one general requirement for this growth: as long as the horizon area is already sufficiently large with respect to r^2, its growth can never be less than $8\pi mr$—independent of the exact nature of the hole [9]. Thus in this process, $\Delta A \geq 8\pi mr$. By conjecture, though, $\Delta S = \eta \Delta A$. So if there were some principle to fix a minimum ΔS, one could start to imagine a means for bounding η. At this point, Bekenstein takes the following position. Upon the particle's crossing the horizon, any amount of information concerning that particle (composition, color, *etc.*) may be lost; the amount will just depend on how much was known beforehand. At the very least, though, the question of the particle's *existence* can no longer be answered by an outside observer. This he justifies by pointing out that under no circumstances can communication take place across the horizon. Thus, associated with the hypothesis of existence, one must now use the probabilities $p_{\rm yes} = p_{\rm no} = \frac{1}{2}$, whereas before the particle's capture one necessarily had to use $p_{\rm yes} = 1$ and $p_{\rm no} = 0$. In other words, this process gives a statistical information loss (in the Shannon sense) of at least $-\frac{1}{2} ln \frac{1}{2} - \frac{1}{2} ln \frac{1}{2} = ln 2$, *i.e.* one bit. Equating this statistical information loss *à la* Jaynes [13] with the ΔS for a minimal area increase, the bound $\eta \geq (ln 2)/8\pi mr$ follows. The direction of the inequality comes from the fact that the $8\pi mr$ growth can actually be attained with the correct orbit while the possibility of the attainment of the $ln 2$ is clearly more debatable.

Now, however, it should be apparent that eventually quantum theory will have to be invoked. Classical theory cannot complete the reasoning: by making the particle's mass or radius arbitrarily small, one can make the lower bound on η arbitrarily large. First consider the case in which the particle's mass m is fixed at the outset. Here relativistic quantum theory denies the possibility of $\eta \rightarrow \infty$ by demanding that particles be represented by wave packets, and moreover, that wave packets of "width" smaller than the particle's Compton wavelength $1/m$ be built not only of positive energy components but also negative ones, *i.e.* antiparticles. (For simplicity it is assumed that the particle's gravitational radius $2m$ is smaller than its Compton wavelength.) So to insure that a packet does indeed contain one and only one particle, and thus contribute only the minimal $ln 2$ to the "existence" information loss, its width can be no smaller than a Compton wavelength. Alternatively, one could consider the case that r is fixed at the outset and take the limit that $m \rightarrow 0$. But here again the same problem appears; r becomes smaller than $1/m$. Therefore, Bekenstein makes the claim that at minimum $\Delta A \approx 8\pi m(1/m) = 8\pi$ so that consequently $\eta \geq .028$.

This original scheme of Bekenstein's is reviewed in such detail because, although two of its points must be severely criticized (with hindsight of course), it does after all provide the backbone for the sought after implementation of Landauer's principle. So, on to the first criticism; the second criticism is saved for Section III.

First and foremost one must ask just how sound the justification really is for the one-bit information-loss minimum. What is it that singles out the question of existence as more infallibly applicable than any of a number of others that are just as clearly suggested by the problem? For instance: questions of existence aside, does the particle still possess a position in spacetime after crossing the horizon? Does it still possess a four-momentum? Just as with the question of existence, these too cannot be answered by an external observer. Without a further *philosophical* criterion for tying all these (and perhaps other) questions together, one might argue that the *true* minimum bit loss is much greater than that suggested by Bekenstein. The point to be made here is that perhaps Bekenstein's argument is not as physical as it should be and that perhaps the search for the "real" information loss is not quite the correct route to take in fixing a minimum ΔS.

This is where Landauer's principle enters the scene. Consider again the same scenario of a particle free falling into a sufficiently large black hole. (Here we analyze this situation from the perspective of the black hole; elsewhere [14] we develop it from an outsider's point of view.) By the local gravitational field laws, the horizon area must increase by at least $8\pi mr$ in the absorption of the particle. Alternatively however, by the entropic version of Landauer's principle, the black-hole entropy must also adjust accordingly; for the black hole, again given only its local interaction with the particle, cannot *know* that the particle was not part of some larger information bearing signal—for instance, a signal representing the results of a Maxwellian demon's measurements on an N-box Szilard engine [2-5]. Thus one might as well act as if the given particle is indeed a component in some larger signal. Without loss of generality for the present argument, it is most convenient to assume this hypothetical signal to be of a "time delay between particle" type: conceptual boxes occupied by particles might represent the 1's in a binary string, while conceptual boxes occupied by void might represent the 0's. For example:

 $\leftrightarrow$ 1101000101

As can be seen by the varied positions of the particles in the boxes of this illustration, there is nothing in this coding scheme that requires the conceptual-box size to be determined by the actual particle size. This scheme allows for the wisest use of the framework developed earlier since the particles themselves, *i.e.* the only signal components capable of increasing the black-hole area, actually stand for bits.

Now a most useful question arises. Agreeing that the black-hole entropy *will* adjust when a particle is absorbed, how much *must* it adjust? To this end one must recognize that if the particle is part of some larger signal, then by necessity it represents in this scheme a digit in a binary string x, say, of some fixed (although perhaps unspecified) length N, for instance. This says nothing more than that before one can calculate an entropy, one must pick a definite system. Hence one can invoke a well known theorem of algorithmic information theory to write at least formally [6]:

$$c_1 \leq K(x) \leq N + K(N) + c_2,$$

where moreover (1)

$$K(N) \leq 2\, log\, N + c_3.$$

(Here and elsewhere *log* is taken to mean the base 2 logarithm and all c_i, $i = 1, 2, \ldots$, are positive integer constants independent of N and x.) This makes clear that, by Zurek's proposal, if the black hole were to absorb the whole signal x, its entropy must increase by anywhere from c_1 to $N + 2\, log\, N + c_4$ bits, depending on the true complexity of x. Since this situation is completely specified there is no statistical contribution H to Zurek's *physical entropy* for the signal, *i.e.* when discussing the signal x, the point of view of a completely informed Maxwell demon is used. If the black hole were to ultimately absorb only part of the signal, then its entropy need only increase to an appropriately lesser extent. This leads to the crucial point. If the black hole cannot *know* whether the given particle is part of some larger signal, then it certainly cannot *know* in advance the algorithmic complexity of the whole signal or even whether the whole signal will be eventually absorbed. Therefore, to cover all bases and ultimately insure Landauer's principle, the black-hole entropy had better adjust in a way appropriate for a maximum information content signal of some unspecified but fixed length N.

With the last point made, all that is left to complete the present line of reasoning is a clarification of the phrase "adjust in a way appropriate for a maximum information content signal." Then the method for estimating η based purely on Landauer's principle will be at hand. Here again a standard theorem of algorithmic information theory will provide the needed tool. For it turns out that maximally complex strings have an essentially fixed distribution of 0's and 1's. Let S_x denote the number of 1's in x. Then there exists a c_5 and c_6 such that

$$K(x) > N + c_5 \text{ implies } |2S_x - N| \leq c_6\sqrt{N}. \quad (2)$$

This theorem, first shown by Martin-Löf [15], is one among many reasons strings of maximal complexity are called "random"; random strings have an approximately equal number of 0's and 1's. Hence, assuming that the particles in the hypothetical signal are noninteracting, the phrase "adjust in a way appropriate ..." can only mean:

$$\Delta A \rightarrow \left[(N/2) + c_7\sqrt{N}\right]8\pi mr$$

while

$$\Delta S \rightarrow \left[N + 2\,log\,N + c_4\right]ln\,2. \quad (3)$$

Although it is of no real importance, the plus sign in the first of these is chosen so that the bound on η cannot be overestimated. Since there is no way of fixing the constants in these prescriptions, one need merely take the limit of large N for the final estimate:

$$\eta \geq \lim_{N\to\infty} \frac{\Delta S}{\Delta A} = \frac{ln\,2}{4\pi mr}. \quad (4)$$

That is to say, each particle absorbed by the black hole must contribute at least two bits of entropy.

A few comments are still in order for the reasoning behind the inequality in (4). First, as in the case with Bekenstein's estimate, there is nothing that requires ΔS to be as small as argued here; Landauer's principle can also clearly be salvaged by a more "irreversible" reaction. Second though, and of perhaps more interest, is the possibility that $8\pi mr$ may not be the absolute minimum area growth for the erasure of a bit. Instead of allowing the signal to free fall past the event horizon, one might imagine lowering it down with a rope, say, and so effectively reducing its mass by extracting work from the process. Bekenstein [8] has argued that ultimately before the particle can be absorbed it must be released from the rope and hence be in free fall; this would force the $8\pi mr$ lower bound to hold always. Whether this is a valid point is not clear, especially since in the end the quantum mechanical nature of matter must be taken into account. One thing, however, is for sure. If the signal is lowered down with a rope, the container housing the signal will be accelerated in the black-hole spacetime and so will feel a quantum field theoretic acceleration radiation [16] whose buoyancy-giving pressure would have to be taken into account. At this point the game of estimating η from simple principles (such as Landauer's) would be lost, not least of all since η itself can be extracted from the radiation calculation [11,16].

Notice that so far the only thing that distinguishes estimate (4) from Bekenstein's original is a factor of two. The remainder of the discrepancy is found in the second criticism of that work.

III. The Quantum Analog to $\Delta A \geq 8\pi mr$

Bekenstein argues that the product of a particle's mass and radius, mr, can be roughly no smaller than unity because semiclassically r cannot be smaller than $1/m$ without "pair production." But how well-founded is this argument really? Answering the question of how the $1/m$ criterion arises rigorously leads to the next step in the estimate of η.

Ultimately this phenomenon comes from the fact that in a (special) relativistic quantum theory the inner product must be Lorentz invariant. This can be seen as follows. Consider the inner product in momentum space for positive energy solutions to the Klein-Gordon equation describing neutral mass-m spin-zero particles [17]:

$$\langle\psi\,|\,\phi\rangle = \int \frac{d^3p}{\sqrt{p^2+m^2}}\,\psi^*(\mathbf{p})\,\phi(\mathbf{p}). \quad (5)$$

The extra factor of $\left(p^2+m^2\right)^{-1/2}$ accompanying the standard d^3p in this integral's measure allows this inner product to be invariant under boosts. Now in nonrelativistic quantum mechanics the momentum and position operators ($\hat{p}_j$ and $\hat{x}_j$, $j = 1,2,$ 3) are represented in $\mathbf{p}$-space by

$$\hat{p}_j = p_j \quad \text{and} \quad \hat{x}_j = i\frac{\partial}{\partial p_j}. \quad (6)$$

The important thing to note here, though, is that with respect to the inner product (5), the operator $\hat{x}_j$ is *not* hermitian. In other words, $\hat{x}_j$ is not an observable in the relativistic theory. The correct counterpart to $\hat{x}_j$ in the relativistic theory is the Newton-Wigner relativistic position operator $\hat{q}_j$ [18] given in $\mathbf{p}$-space by

$$\hat{q}_j = i\frac{\partial}{\partial p_j} - \frac{i}{2}\frac{p_j}{p^2+m^2}. \quad (7)$$

Not only is this operator hermitian with respect to (5), but it is also uniquely justified [18] in the sense that its eigenfunctions satisfy a reasonable criterion of being "localized." Moreover, as one would hope,

$$[\hat{q}_j, \hat{q}_k] = 0 \quad \text{and} \quad [\hat{q}_j, \hat{p}_k] = i\,\delta_{jk}. \tag{8}$$

Finally Newton and Wigner point out that expressions (5), (7), and (8) still hold even in the case that $m = 0$.

This much formalism introduced, the $1/m$ criterion now follows quite easily. One need only verify that up to normalization and phase the eigenfunction of $\hat{\boldsymbol{q}}$ with eigenvalue $\boldsymbol{y}$ in its $\hat{\boldsymbol{p}}$-space representation is

$$\phi(\boldsymbol{p}) = \left(p^2 + m^2\right)^{1/4} e^{-i\boldsymbol{p}\cdot\boldsymbol{y}}. \tag{9}$$

This is given in its $\hat{\boldsymbol{x}}$-space representation by

$$\begin{aligned} \psi(\boldsymbol{x}) &= (\text{const}) \int \frac{d^3p}{\sqrt{p^2+m^2}}\,\phi(\boldsymbol{p})\,e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \\ &= \left(m/r\right)^{5/4} H^{(1)}_{5/4}(imr), \end{aligned} \tag{10}$$

where $r = |\boldsymbol{x} - \boldsymbol{y}|$ and $H^{(1)}_{5/4}$ is the order-5/4 Hankel function of the first kind. Clearly this eigenfunction is not the Dirac delta $\delta(\boldsymbol{x} - \boldsymbol{y})$ that one is used to from nonrelativistic quantum mechanics; it only blows up as $r^{-5/2}$ when $r \to 0$ and falls off as e^{-mr} when $r \to \infty$. Because of the exponential decay with large r, one may associate a characteristic width of $1/m$ with $\psi(\boldsymbol{x})$. This last point gives rise to the taking of $1/m$ as the approximate minimum wave packet "width."

Now the points of contention with the original analysis can be faced; these are twofold. First and perhaps most basic, why should the characteristic width of the $\hat{\boldsymbol{x}}$*-space representation* of an eigenket $|\boldsymbol{q}\rangle$ have anything to do with a quantum mechanical analog to the requirement $\Delta A \geq 8\pi m r$? The operator $\hat{\boldsymbol{x}}$ is not even an observable in the relativistic quantum theory that gave rise to these considerations; so too it must be the case with this "width." Furthermore, even with that point aside for the moment, one is still left with the following question. If a semiclassical analysis is used for the classical r, then should not that also be the case for the classical m (*i.e.* not the m appearing in the quantum theory above)? The correct analog to a "rest mass" for a wave packet would appear to be something like $\langle \hat{E} \rangle$ the expectation value of $\hat{E}$ the relativistic energy operator or even possibly the root mean square. This idea, though, leads to a definite problem here. Recall that in $\hat{\boldsymbol{p}}$-space $\hat{E}$ is given by

$$\hat{E} = \sqrt{p^2 + m^2}\,. \tag{11}$$

With this one clearly has for the minimum width packet (9):

$$\langle \hat{E} \rangle = (\text{const}) \int d^3p\,\sqrt{p^2+m^2} \;\to\; \infty. \tag{12}$$

In other words, $\langle \hat{E} \rangle$ is simply not defined for this packet. Therefore within the present context these two ideas—seemingly required for consistency of method—cannot be of much use.

The points made in the last paragraph taken together lead to the following conjecture. Since the requirement $\Delta A \geq 8\pi m r$ ultimately arises from a classical equation of motion, one can invoke the correspondence principle [19] to assert that a more accurate quantum analog would be

$$\Delta A \;\geq\; 8\pi \; min\Big(\langle \hat{E} \rangle \langle \hat{q} \rangle\Big) \tag{13}$$

where the *min* stands for taking the minimum over all possible wave packets $\phi(\boldsymbol{p})$ subject only to the normalization condition

$$\int \frac{d^3p}{\sqrt{p^2+m^2}}\,\phi^*(\boldsymbol{p})\,\phi(\boldsymbol{p}) \;=\; 1, \tag{14}$$

and $\hat{q}^2 = \hat{q}_1^2 + \hat{q}_2^2 + \hat{q}_3^2$.

Equation (13) may be taken as the starting point for returning to the considerations concerning Landauer's principle. Before starting, though, one can go one step further. Since no particular mass m in this relativistic quantum theory (not even $m = 0$) is singled out as distinct from any other, one might as well take the limit $m \to 0$ in (13) at the outset. This stands the chance of simplifying things greatly. The remainder of this section is devoted to explicitly calculating a numerical value for $min(\langle \hat{E} \rangle \langle \hat{q} \rangle)$ for one-dimensional wave packets with $m = 0$.

For simplicity in the calculation to come, note that $\langle \hat{E} \rangle$ and $\langle \hat{q} \rangle$ may be written in a formal representation free way as $\langle \phi | \hat{E} | \phi \rangle$ and $\langle \phi | \hat{q} | \phi \rangle$, respectively. Similarly the normalization condition (14) can be written as $\langle \phi | \phi \rangle = 1$. Now in order for $\langle \hat{E} \rangle \langle \hat{q} \rangle$ to be a minimum it must at least be stationary with respect to variations in $\langle \phi |$. Here it is assumed that variations of $\langle \phi |$ are independent of $|\phi\rangle$. Introducing a Lagrange multiplier λ' for the constraint (14) and performing the variation, one obtains the eigenvalue equation

$$\Big(\langle \hat{q} \rangle \hat{E} + \langle \hat{E} \rangle \hat{q}\Big) |\phi\rangle \;=\; \lambda' |\phi\rangle. \tag{15}$$

Acting back on this equation with $\langle \phi |$, one trivially finds that

$$\lambda' = 2\langle \widehat{E} \rangle \langle \widehat{q} \rangle. \tag{16}$$

Therefore, finding the minimum eigenvalue of (15) is enough to solve the problem at hand.

Note that in one dimension with m set to zero the analogues to $\widehat{E}$ and $\widehat{q}$ are $|\,\widehat{p}_1\,|$ and $|\,\widehat{q}_1\,|$, respectively. Using this and the fact that the computation of the minimum λ' can be simplified even further with the assumption $\langle \widehat{E} \rangle = \langle \widehat{q} \rangle$, one obtains the eigenvalue problem

$$\tfrac{1}{2}\Big(|\,\widehat{p}_1\,| + |\,\widehat{q}_1\,| \Big) |\,\phi\rangle = \lambda\, |\,\phi\rangle. \tag{17}$$

The ultimate answer to the mr-analogue problem will then be given by λ^2_{min}.

The only real difficulty now is in interpreting the operators $|\,\widehat{p}_1\,|$ and $|\,\widehat{q}_1\,|$. How does one take the absolute value of a differential operator? The lack of an answer to this question preempts any possibility of solving (17) directly. This problem may however be sidestepped if one is willing to use approximation techniques. In particular, first consider the eigenvalue problem

$$\tfrac{1}{2}\Big(\widehat{p}_1^{\,2} + \widehat{q}_1^{\,2} \Big) |\,\phi\rangle = \mu\, |\,\phi\rangle. \tag{18}$$

Because the commutation relations (8) are satisfied by $\widehat{p}_1$ and $\widehat{q}_1$, this is just a generalized harmonic oscillator problem with eigenvalues $\mu = (n + \frac{1}{2})$, $n = 0, 1, 2, \ldots$ [19]. Of course one really should be careful to check that all this still holds with respect to the relativistic inner product (5), but that task is readily tractable. Writing the solutions to (18) as $|\,n\rangle$, it is not difficult to verify that

$$\langle p_1 \,|\, n\rangle = \pi^{-\frac{1}{4}}\, 2^{-\frac{n}{2}}\, (n!)^{-\frac{1}{2}}\, |\,p_1\,|^{\frac{1}{2}}\, e^{-\frac{1}{2}p_1^2} H_n(p_1)$$

and (19)

$$\langle q_1 \,|\, n\rangle = i^n\, \pi^{-\frac{1}{4}}\, 2^{-\frac{n}{2}}\, (n!)^{-\frac{1}{2}}\, e^{-\frac{1}{2}q_1^2} H_n(-q_1)$$

are their momentum and q_1-space representations respectively. With these in hand the means for getting around the problem mentioned at the beginning of the paragraph is immediate. Expand the trial solutions to (17) in terms of these harmonic oscillator eigenstates and use the linear variational method [19] to estimate λ. The variational method has the advantage that it only makes use of the matrix elements

$$\langle n \,|\, \tfrac{1}{2}(\,|\,\widehat{p}_1\,| + |\,\widehat{q}_1\,|\,)\,|\, m\rangle = \tfrac{1}{2}\langle n|\ |\,\widehat{p}_1\,|\ |m\rangle + \tfrac{1}{2}\langle n|\ |\,\widehat{q}_1\,|\ |m\rangle.$$

If each of these terms is evaluated in the appropriate representation, no problem in interpreting the $|\ |$ will arise. Thus this is clearly the way to tackle the problem.

Calling the matrix elements to the operator in (17) H_{nm}, $n, m = 0, 1, \ldots, N$, the linear variational method dictates that λ_{min} be approximated by its lowest eigenvalue. This has been calculated with the help of the eigenvalue package in *Mathematica*™ for N up to 16. (Actually with a little finesse one can show that the $N = 16$ problem reduces to the evaluation of the eigenvalues of the 5×5 matrix whose elements are produced from $|\,n\rangle$ for $n = 0, 4, 8, 12, 16$.) In this approximation λ_{min} is found to be given by .5523.

IV. Conclusion

The results of Sections II and III taken together give the final numerical estimate for the black-hole entropy to area ratio as

$$\eta \geq \frac{\ln 2}{4\pi(.5523)^2} \approx .181. \tag{20}$$

The ratio between the established value for η of 1/4 and this lower bound is 1.38. This contrasts with 8.93 for the same ratio only using Bekenstein's original "information theoretic" bound.

This result argues well that the entropic formulation of Landauer's principle is valid outside its first arena of justification even though its general validity is still possibly far from confirmed. Moreover it demonstrates that the principle may be of some predictive efficacy. For if η had not been known beforehand, one would not have been too far off the mark with this general and indeed simple argument. Finally, viewing Landauer's principle as just what it is —a thermodynamic accounting rule—one is led back (but with perhaps new insight) to the magical question of how it is that spacetime structure knows the Second Law of Thermodynamics. Black holes can erase information in a detailed sense.

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